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On Minimal H-Sets*

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1. Introduction

In this paper a question in approximation theory which was related to the author by Professor L. Collatz will be answered. To some degree this question was also posed by Rivlin and Shapiro [4, p. 697] and is closely related to some work of Brosowski [1].

Following the development of Collatz [2], we let B be a subset of R^n and define $C(B)$ to be the space of all continuous real-valued functions defined on B for which $\| f \| = \sup\{| f(x)| : x \in B\}$ is finite. Next, let $W = \{a_0 + a_1x_1 + \cdots + a_nx_n\}$ where a_i is real for all i and $(x_1, ..., x_n) \in \mathbf{R}^n$. The approximation theory problem in which one is then interested is that of approximating functions of $C(B)$ by W. In this setting certain subsets of B, called H-sets, play an important role due to the existence of an "Inclusion Theorem" [3, p. 418–420].

The set $M = M_1 \cup M_2$ is said to be an H-set for W provided $M_1, M_2 \subset B$, $M_1 \cap M_2 = \emptyset$ and there exists no pair $w_1, w_2 \in W$ satisfying

$$\begin{aligned} w_1 - w_2 &> 0 \qquad \text{on } M_1, \\ w_1 - w_2 &< 0 \qquad \text{on } M_2. \end{aligned}$$

M is said to be a minimal H-set provided $N_1 \subset M_1$ and $N_2 \subset M_2$ with at least one containment proper implies $N = N_1 \cup N_2$ is not an H-set for W. For a discussion of H-sets, see [2, 3].

The question that we shall answer here is that of classifying and enumerating all possible types of minimal H-sets corresponding to W (for fixed n). It was shown by Collatz [2] that when $n = 1, 2, 3$ (n = dimension of $\mathbf{R}^n$) there exist 1, 3, and 5 basic minimal H-sets, respectively.

In the work of Rivlin and Shapiro [4] and of Brosowski [1] the term

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primitive extremal signature is equivalent to minimal H-set (for our setting). Brosowski studied the question of classifying all minimal H-sets corresponding to $W = \{a_0 + \cdots + a_n x_n\}$ and gave an answer to this question in terms of classes (k, j) [1, pp. 398–399].

We shall approach this problem from a somewhat different geometrical view than that taken by Brosowski and thus we shall arrive at a "different" classification. Also, we shall use our classification to count the total number of basic minimal H-sets corresponding to W. Relating this work to [1], we will have thus shown that each class (k, j) corresponds to exactly one basic minimal H-set.

2. Main Results

We shall begin this section by stating a characterization for H-sets (or extremal signatures) which was given by Rivlin and Shapiro [4, p. 696].

THEOREM 1. *$M = M_1 \cup M_2$ is an H-set for W if and only if*

$$H(M_1) \cap H(M_2) \neq \emptyset,$$

where $H(M)$ denotes the convex hull of M.

In what follows we shall prove a refinement of this theorem for the special case of minimal H-sets and from this obtain our desired results. We begin by remarking that we shall *only count* minimal H-sets for which *cardinality* $M_1 \geqslant$ *cardinality* M_2 due to the obvious symmetry that exists.

At the outset let us make some rather obvious observations.

LEMMA 1. *In $\mathbf{R}^n$, $M = M_1 \cup M_2$ being a minimal H-set for W implies*

(i) *Both M_1 and M_2 are finite sets of $\leqslant n + 1$ elements.*

(ii) *If $M_1 = \{p_1, \ldots, p_m\}$ then $p_2 - p_1, \ldots, p_m - p_1$ forms a linearly independent set of vectors in $\mathbf{R}^n$; that is, $H(M_1)$ is an $m - 1$ simplex. (Likewise, for M_2).*

Proof. (i) This follows immediately from the definition of minimality and Carathéodory's theorem.

(*ii*) Suppose $M = M_1 \cup M_2$ is a minimal H-set for W and $M_1 = \{p_1, \ldots, p_m\}$ with $p_2 - p_1, \ldots, p_m - p_1$ a linearly dependent set of vectors. Let $p \in H(M_1) \cap H(M_2)$. Then

$$p - p_1 \in [H(M_1) - p_1] \cap [H(M_2) - p_1] = H(M_1 - p_1) \cap H(M_2 - p_1).$$

Now $M_1 - p_1 = \{0, p_2 - p_1, \ldots, p_m - p_1\}$ is contained in a subspace of $\mathbf{R}^n$ of dimension at most $m-2$. Thus any point in the convex hull of this set may be represented by a convex combination involving at most $m-1$ of these points by Catathéodory's theorem. Let i_0 be the first index of the representation of $p-p_1$ for which the convex constant is zero. Then

$$p-p_1=\sum_{i=1, i\neq i_0}^{m} \lambda_i(p_i-p_1), \lambda_i \geqslant 0, \sum_{i=1, i=i_0}^{m} \lambda_i = 1.$$

That is,

$$p=\sum_{i=1, i\neq i_0}^{m} \lambda_i p_i$$

implying that $p \in H(\tilde{M}_1) \cap H(M_2)$, where $\tilde{M}_1 = M_1 \sim \{p_{i_0}\}$ which contradicts the hypothesis that $M = M_1 \cup M_2$ is a minimal H-set.

Thus we have reduced our search for minimal H-sets to a search involving simplices. We now proceed to study how simplices may intersect. We shall use the word simplex to mean a nondegenerate simplex.

LEMMA 2. *Let $M_1 = \{p_1, ..., p_m\}$ and $M_2 = \{q_1, ..., q_k\}$ be the vertices of a $m-1$ simplex S_1 and a $k-1$ simplex S_2, respectively, where $M_1, M_2 \subset R^n$ with $n \geqslant m+k-2$ and $m \geqslant k$. If $M_1 \cap M_2 = \emptyset$ and $S_1 \cap S_2$ consists of either a single point which is on a face of one of the simplices or at least two points, then $M = M_1 \cup M_2$ is not a minimal H-set.*

Proof. Let $S_1 \cap S_2 = \{p\}$ where without loss of generality we assume $p = \sum_{i=2}^{m} \lambda_i p_i$, $\lambda_i \geqslant 0$, $\sum_{i=2}^{m} \lambda_i = 1$ (p belongs to the face of S_1 opposite p_1). But from this representation it is clear that $p \in H(\{p_2, ..., p_m\}) \cap H(M_2)$ implying that $M = M_1 \cup M_2$ is not a minimal H-set.

Next, let us assume that $S_1 \cap S_2$ contains at least two points p, q, each of which is an interior point for both S_1 and S_2 (for otherwise we could use the above proof to see the M is not a minimal H-set). Thus the line joining these points, $\lambda p + (1-\lambda)q$, $0 \leqslant \lambda \leqslant 1$, is in the interior of both S_1 and S_2. But now for a proper choice of $\lambda > 1$ one obtains a point of $S_1 \cap S_2$ which is on a face of at least one of these simplices. Thus we may apply the previous argument to conclude M is not a minimal H-set.

Using these lemmas, we can now prove our desired refinement of Theorem 1.

THEOREM 2. *$M = M_1 \cup M_2$* (card $M_1 \geqslant$ card M_2) *is a minimal H-set for W if and only if $M_1 = \{p_1, ..., p_m\}$ and $M_2 = \{q_1, ..., q_k\}$ form a $m-1$ simplex S_1 and a $k-1$ simplex S_2, respectively, $M_1 \cap M_2 = \emptyset$ and $S_1 \cap S_2$ intersects in a single point interior to both S_1 and S_2.*

Proof. (only if) This follows from Lemma 1 and Lemma 2.

(if) To see this part of the Theorem we begin by noting that M is an H-set by Theorem 1. To see that it is minimal we must only note that if one omits a single point from either M_1 or M_2 then the new simplices formed will no longer intersect. This can be easily seen by looking at the convex expressions of the points involved.

Our next task is to study the above requirement on simplices. We begin by showing that this phenomenon does not occur too often.

LEMMA 3. *If* $M_1 = \{p_1, ..., p_m\}$ *and* $M_2 = \{q_1, ..., q_k\}$ *are the vertices for a* $m - 1$ *simplex* S_1 *and a* $k - 1$ *simplex* S_2, *respectively, with* $M_1 \cap M_2 = \emptyset$, $M_1 \cup M_2 \subset R^n$, $m \geqslant k$ *and* $n < m + k - 2$, *then* $S_1 \cap S_2$ *cannot consist of only one point which is interior to both* $S_1 \cap S_2$.

Proof. Suppose $S_1 \cap S_2$ consists of only a single point p interior to each simplex. Then there exist open balls (open in R^{m-1} and R^{k-1}, respectively) B_1, B_2 centered at p such that $B_1 \subset S_1$, $B_2 \subset S_2$, and $B_1 \cap B_2 = \{p\}$. But this is impossible since we would now have $m + k - 2$ linearly independent (actually orthogonal) vectors in R^n where $n < m + k - 2$.

LEMMA 4. *If there exists* j *minimal H-sets in* R^{n-1} *then there exists* $j + [n/2] + 1$ *minimal H-sets in* R^n, *where* $[a]$ *denotes the greatest integer in* a.

Proof. We begin by noting that any minimal H-set in R^{n-1} is also a minimal H-set in R^n. This follows immediately from Theorem 2. Thus any additional minimal H-sets in R^n (but not in R^{n-1}) must be such that no $n - 1$ dimensional hyperplane of R^n contains the union of their convex hulls. Since in this case we would have already counted this configuration while counting the minimal H-sets of R^{n-1}. Using Lemma 3, we see that we need only consider those cases where $m + k - 2 = n$, $m \geqslant k \geqslant 1$. But here a straightforward enumeration shows that we can have at most $[n/2] + 1$ additional minimal H-sets.

To see that each of these possibilities does represent a minimal H-set it is only necessary to construct (corresponding to a fixed m and k satisfying $m + k - 2 = n$, $M \geqslant k \geqslant 1$) sets $M_1 = \{p_1, ..., p_m\}$ and $M_2 = \{q_1, ..., q_k\}$ both contained in R^n such that $H(M_1)$ is an $m - 1$ simplex and $H(M_2)$ is an $k - 1$ simplex, $M_1 \cap M_2 = \emptyset$ and $H(M_1) \cap H(M_2)$ is a single point interior to each of these simplices. Such a construction can be obtained by writing $R^n = R^{m-1} \oplus R^{k-1}$ and constructing an $m - 1$ simplex in R^{m-1} with barycenter at the origin of R^{m-1} and doing likewise in R^{k-1}. Then by embedding each simplex in R^n in the obvious manner we shall have our desired configuration. (The existence of these minimal H-sets was also given by Brosowski.)

THEOREM 3. *In R^n there are precisely $h(n)$ minimal H-sets* (card $M_1 \geqslant$ card M_2) *where*

$$h(n) = \begin{cases} k^2 + 2k, & \text{if } n = 2k, \\ k^2 + 3k + 1, & \text{if } n = 2k + 1. \end{cases}$$

Proof.

$$\sum_{j=1}^{n} \left(\left[\frac{j}{2} \right] + 1 \right) = \begin{cases} k^2 + 2k, & \text{if } n = 2k \\ k^2 + 3k + 1, & \text{if } n = 2k + 1. \end{cases}$$

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